

Product of the Permutable Subgroups

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Abstract

In this paper we check the permutable product of supersoluble subgroups. And the end, we obtain sufficient conditions for permutable products of finite supersoluble groups to be supersoluble.

Keywords: supersoluble; product groups; finite groups

1. Introduction

Two subgroups A and B of a group G are called permutable if every subgroup X of A is permutable with every subgroup Y of B , i.e., XY is a subgroup of G . In this case, if $G=AB$ we say that G is the permutable product of the subgroups A and B .

Throughout the paper only finite groups are considered.

It is known that products of supersoluble groups are not

supersoluble in general. Consequently the problem of finding sufficient

conditions to assure that a product of two supersoluble groups is

supersoluble has received considerable attention. In this paper we prove the following theorem:

If the group G is product of the supersoluble subgroups A and B . Then by what condition, G is supersoluble?

Main results The following four lemmas are needed to prove Theorem 1.

Lemma 1 [4, Theorem 2]. If $G=AB$ is the mutually permutable product of the supersoluble subgroups A and B , then G is soluble.

Lemma 2 ([12]). Let $G=AB$ be the mutually permutable product of the supersoluble subgroups A and B . Then, either G is supersoluble or $NA < G$ and $NB < G$ for every minimal normal subgroup N of G .

Lemma 3 ([12]). Let $G=AB$ be the mutually permutable product of the subgroups A and B and let N be any minimal normal subgroup of G . Then either

$N \cap A = N \cap B = 1$ or $N = (N \cap A)(N \cap B)$.

Lemma 4 [8, Lemma A.13.6]. Let G be a group, and N a minimal normal subgroup of G such that $|N|=pn$, where p is a prime and $n>1$. Denote $C=CG(N)$ and assume that G/C is supersoluble. Then, if Q/C is a subgroup of G/C containing $Op'(G/C)$, we have that Q is normal in G and $N=\prod_{i=1}^t N_i$, where N_i are non-cyclic minimal normal subgroups of NQ for $i=1, \dots, t$.

Proof of Main Theorem.

Let $G=AB$ be the mutually permutable product of the supersoluble subgroups A and B , with $\text{Core}_G(A \cap B)=1$, and suppose that G has been chosen minimal such that its supersoluble residual GU is non-trivial. Let N be a minimal normal subgroup of G contained in GU . Note that N is an elementary abelian p -group for some prime p . Applying Lemma 2, we have that both NA and NB are proper subgroups of G . Moreover, using Lemma 3, we have that either $N=(N \cap A)(N \cap B)$ or $N \cap A = N \cap B = 1$. Assume first that $N=(N \cap A)(N \cap B)$.

(i) If $N \cap A = 1$, then N is cyclic. Assume that $N \cap A = 1$. It follows that N is contained in B . Let N_0 be a non-trivial cyclic subgroup of N . Since AN_0 is a subgroup of G , we have that $N_0 = AN_0 \cap N$ is a normal subgroup of AN_0 . Hence every cyclic subgroup of N is normalised by A . Now let N_1 be a minimal normal subgroup of B contained in N . Since B is supersoluble, it follows

That N_1 is cyclic and thus normalised by A . Hence N_1 is a normal subgroup of G . The minimality of N implies that $N=N_1$ and consequently N is cyclic.

(ii) $N \cap A = 1$ and $N \cap B = 1$. On the contrary, assume that $N \cap A = 1$. From (i), we know that N is cyclic. Moreover, N is contained in B . Hence $AN \cap B = (A \cap B)N$. Let $L = \text{Core}_G(A \cap B)N$. Clearly, N is contained in

Land $L=L\cap((A\cap B)N)=(L\cap A\cap B)N$. It is clear that $G/L=(AL/L)(BL/L)$ is a mutually permutable product of AL/L and BL/L such that $\text{Core } G/L((AL/L)\cap(BL/L))=1$. By the minimality of G , it follows that G/L is supersoluble. On the other hand, since N is cyclic, we have that $G/\text{CG}(N)$ is abelian. Hence $G/\text{CL}(N)$ is supersoluble and $\text{GUCL}(N)=C$. Note that $C=N\times(C\cap A\cap B)$. Therefore $C\cap A\cap B$ contains a Hall p' -subgroup of C . Since $\text{Core } G(A\cap B)=1$ and $\text{Op}'(C)$ is a normal subgroup of G contained in $C\cap A\cap B$, we have that $\text{Op}'(C)=1$. Moreover, $C'=(C\cap A\cap B)'$ is a normal subgroup of G contained in $A\cap B$. Consequently, $C'=1$ and C is an abelian p -group. In particular, GU is abelian and thus GU is complemented in G by a supersoluble normalizer D which is also a supersoluble projector of G , by [8, Theorems V.4.2 and V.5.18]. Since N is cyclic, we know that N is central with respect to the saturated formation of all supersoluble groups. By [8, Theorem V.3.2.e], D covers N and thus N is contained in D . It follows $ND\cap\text{GU}=1$, a contradiction.

(iii) Either $N=N\cap A$ or $N=N\cap B$. If we have $N=N\cap A=N\cap B$, then N is contained in $A\cap B$, contradicting the fact that $\text{Core } G(A\cap B)=1$. We may assume without loss of generality that $N\cap A=N$.

(iv) AN and BN are both supersoluble. Since $N=(N\cap A)(N\cap B)$ and $N=N\cap A$, it follows that $N\cap B$ is not contained in $N\cap A$. Let n be any element of $N\cap B$ such that $n\notin N\cap A$, and write $N_0=\langle n \rangle$. Note that AN_0 is a subgroup of G , and $AN_0\cap N=(N\cap A)N_0$. Therefore $N_0(N\cap A)$ is a normal subgroup of AN_0 , and consequently A normalizes $(A\cap N)N_0$. This yields that $A/\text{CA}(N/N\cap A)$ acts as a power automorphism group on $N/N\cap A$. This means that AN is supersoluble. If $N\cap B=N$, then $BN=B$ is supersoluble. On the contrary, if $N\cap B=N$, we can argue as above and we obtain that BN is supersoluble. Consequently, $\text{ACG}(N)/\text{CG}(N)$ and $\text{BCG}(N)/\text{CG}(N)$ are both abelian groups of exponent dividing $p-1$. But then $G/\text{CG}(N)=(\text{ACG}(N)/\text{CG}(N))(\text{BCG}(N)/\text{CG}(N))$ is a π -group for some set of primes π such that if $q\in\pi$, then q divides $p-1$.

(v) Let B_0 be a Hall π -subgroup of B . Then $AB_0\cap N=A\cap N$.

This follows just by observing that $AB_0\cap N$ is contained in each Hall π' -subgroup of AB_0 and every Hall π' -subgroup of A is a Hall π' -subgroup of AB_0 . Note that $|G/\text{CG}(N)|$ is a π -number and AB_0 contains a Hall π -subgroup of G . Therefore $G=(AB_0)\text{CG}(N)$. But then $A\cap N$ is a normal subgroup of G . The minimality of G yields either $A\cap N=1$ or $A\cap N=N$. This contradicts our assumption $1=N\cap A=N$, and so we cannot have $N=(A\cap N)(B\cap N)$. Thus, by Lemma 3, we may assume $N\cap A=N\cap B=1$. Let $M=\text{Core } G(A\cap B\cap N)$. Then $N\cap M=1$ and G/M is supersoluble by the minimality of G . Again, we reach a contradiction after several steps.

(vi) $M=N$. Suppose that $M=N$. Since G/M is supersoluble, we know that N cannot be cyclic. Let us write $C=\text{CG}(N)$, and consider the quotient group G/C . It is clear that G/C is supersoluble. Let $Q/C=\text{Op}(G/C)$. Since $\text{Op}(G/C)=1$ and $(G/C)'$ is nilpotent, it follows that Q/C is a normal Hall p' -subgroup of G/C . Let Bp' be a Hall p' -subgroup of B . Since $|N|$ divides $|B:A\cap B|$, we have that $(A\cap B)Bp'$ is a proper subgroup of B . Let T be a maximal subgroup of B containing $(A\cap B)Bp'$. Then AT is a maximal subgroup of G and $|G:AT|=p=|B:T|$. If N is not contained in AT , we have $G=(AT)N$ and $AT\cap N=1$. Then $|N|=p$, a contradiction. Therefore N is contained in AT . In particular, the family $S=\{X:X \text{ is a proper subgroup of } B, (A\cap B)Bp'X \text{ and } NAX\}$ is non-empty. Let R be an element of S of minimal order. Observe that AR has p -power index in G and thus ARC/C contains $\text{Op}'(G/C)$. Regarding N as a AR -module over $\text{GF}(p)$, we know, by Lemma 4, that N is a direct sum $N=\bigoplus_{i=1}^n N_i$, where N_i is an

irreducible AR -module whose dimension is greater than 1, for all $i\in\{1,\dots,t\}$. Assume that $(A\cap B)Bp'=R$. Then $AR=ABp'$ and thus N is contained in A , a contradiction. Therefore $ABp'\cap B=(A\cap B)Bp'$ is a proper subgroup of R . Let S be a maximal subgroup of R containing $(A\cap B)Bp'$. From the minimality of R , we know that N is not contained in AS . Consequently, there exists some $i\in\{1,\dots,t\}$ such that N_i is not contained in AS , which is a maximal subgroup of AR . Hence $AR=(AS)N_i$. Since N_i is a minimal normal subgroup of AR , it follows that $AS\cap N_i=1$ and $|N_i|=|AR:AS|=|R:S|=p$, a contradiction.

(vii) M is an elementary abelian p -group. Note that $M=N(M\cap A)=N(M\cap B)$ and $|M\cap A|=|M\cap B|=|M|/|N|$. Moreover, $A(M\cap B)$ is a subgroup of G such that $A(M\cap B)\cap M=(M\cap A)(M\cap B)$. Hence $(M\cap A)(M\cap B)$ is also a subgroup of G . If $M\cap A=M\cap B$, then $M\cap A$ is a normal subgroup of G contained in $A\cap B$. This implies that $M\cap A=1$ and consequently $M=N$, a contradiction. It yields that $M\cap A=M\cap B$. Next we see that $(M\cap A)(M\cap B)$ is a normal subgroup of G . Since $(M\cap A)(M\cap B)=M\cap A(M\cap B)$, we have that A normalizes $(M\cap A)(M\cap B)$. Similarly, B normalises

$(M\cap A)(M\cap B)$ since $(M\cap A)(M\cap B)=M\cap B(M\cap A)$. This implies normality of $(M\cap A)(M\cap B)$ in G . Let $X=(M\cap A)(M\cap B)$. Since we cannot have $M\cap A=M\cap B$, $M\cap A$ must be strictly contained in X . Thus $X=X\cap M=(X\cap N)(M\cap A)>M\cap A$ gives us $X\cap N=1$. But then $X\cap N=N$, giving NX . Suppose that Q is a Hall p' -subgroup of $M\cap B$. Then QA is a subgroup and so $QA\cap M=Q(M\cap A)$ is also a subgroup which contains Q . Hence, as $|M:M\cap A|=p^k$ for some k , we have that $QM\cap A\cap B$. Thus $QB\cap MM\cap A\cap B$ and similarly $QA\cap MM\cap A\cap B$. Consequently, QM is contained in $M\cap A\cap B$. Since $QM=\text{Op}(M)$, it follows that $\text{Op}(M)$ is a normal subgroup of G contained in $A\cap B$. Hence $\text{Op}(M)=1$, a contradiction, and consequently $Q=1$ and M is a p -group. Hence N is contained in $Z(M)$ and $M=N\times(M\cap A)=N\times(M\cap B)$. Thus $\phi(M)=\phi(M\cap A)=\phi(M\cap B)$ is a normal subgroup of G contained in $A\cap B$. This implies that $\phi(M)=1$ and M is an elementary abelian p -group, as claimed. (viii) Final contradiction. We have from the previous steps that $M\cap A$ is not contained in $M\cap B$ and that $M\cap B$ is not contained in $M\cap A$ because otherwise, since $|M\cap A|=|M\cap B|$, it follows that $M\cap A=M\cap B$ is a normal subgroup of G contained in $A\cap B$. This would imply $M\cap A=M\cap B=1$, and $M=(M\cap A)N=N$. This fact contradicts step (vi).

Let x be an element of $M\cap B$ such that $x\notin M\cap A$. Then $A\langle x \rangle$ is a subgroup of G , and so is $M_0=A\langle x \rangle\cap M=(A\cap M)\langle x \rangle$. Therefore M_0 is an A -invariant subgroup of G . In particular, since $M=(M\cap A)(M\cap B)$, we have that every subgroup of $M/M\cap A$ is A -invariant; that is, $A/\text{CA}(M/M\cap A)$ acts as a group of power automorphisms on $M/M\cap A$. It is clear that $M/M\cap A$ is A -isomorphic to N . Consequently, $A/\text{CA}(N)$ acts as a group of power automorphisms on N . This implies that A normalises each subgroup of N . Analogously, B normalises each subgroup of N . It follows that N is a cyclic group. We argue as in step (ii) above to reach a final contradiction. We have that G/M is supersoluble and M is abelian. Therefore GUM and thus GU is abelian and complemented in G by a supersoluble normaliser, D say, by [8, Theorem V.5.18]. Since N is cyclic, we know that D covers N and thus $\text{NGU}\cap D=1$, a contradiction. Proof of Theorem 2.

Let $M=\text{GU}$ denote the supersoluble residual of G . Theorem 1 yields that $G/\text{Core } G(A\cap B)$ is supersoluble. Therefore M is contained in $\text{Core } G(A\cap B)$. In particular, M is supersoluble. Let $F(M)$ be the Fitting subgroup of M . Since A and B are supersoluble, we have that $[M,A]F(A)\cap MF(M)$ and $[M,B]F(B)\cap MF(M)$. Consequently, $[M,G]$

is contained in $F(M)$. Note now that the chief factors of G between $F(M)$ and A are cyclic, and recall that G/M is supersoluble. Therefore we have that $G/F(M)$ is supersoluble. This implies that $M=F(M)$ and thus M is nilpotent. Consequently, $G/F(G)$ is supersoluble. We now show that $G/F(G)$ is metabelian. We prove first that A' and B' both centralise every chief factor of G . Let H/K be a chief factor of G . If H/K is cyclic, then as G' centralizes H/K , so do A' and B' . Hence we may assume that H/K is a non-cyclic p -chief factor of G for some prime p . Note that we may assume that H is contained in M because G/M is supersoluble and H/K is non-cyclic. To simplify notation, we can consider $K=1$. Since $F(G)$ centralizes H [8, Theorem A.13.8.b], $G/CG(H)$ is supersoluble. Let Ap' be a Hall p' -subgroup of A . By Maschke's theorem [8, Theorem A.11.5], H is a completely reducible Ap' -module and HAp' is supersoluble because H is contained in A . Therefore $Ap'/CAp'(H)$ is abelian of exponent dividing $p-1$. This implies that the primes involved in $|A/CA(H)|$ can only be p or divisors of $p-1$. The same is true for $|B/CB(H)|$. This implies that if p divides $|G/CG(H)|$, then p is the largest prime dividing $|G/CG(H)|$. But since $Op(G/CG(H))=1$ and $G/CG(H)$ is supersoluble, it follows that $G/CG(H)$ must be a p' -group. Consider H as A -module over $GF(p)$. Since $ACG(H)/CG(H)$ is a p' -group, we have that H is a completely reducible A -module and every irreducible A -submodule of H is cyclic. Consequently A' centralizes H , and the same is true for B' . Let now U/V be a chief factor of G . Then $G/CG(U/V)$ is the product of the abelian subgroups $ACG(U/V)/CG(U/V)$ and $BCG(U/V)/CG(U/V)$. By Itô's theorem [9], we have that $G/CG(U/V)$ is metabelian. Since $F(G)$ is the intersection of the centralisers of all chief factors (again by [8, Theorem A.13.8.b]), we can conclude that $G/F(G)$ is metabelian. 3. Final remarks Finally, Theorem 1 enables us to give succinct proofs of earlier results on mutually permutable products.

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